Stabilizing unstable periodic points of one-dimensional nonlinear systems using delayed-feedback signals

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This paper presents a technique which realizes the following operation: a stable orbit in a one-dimensional nonlinear system moves to an already coexisting unstable periodic point and then stabilizes on it. The technique uses two delayed-feedback signals: the first signal destabilizes the nonlinear system such that the orbit wanders about phase space; the second signal stabilizes the wandering orbit onto a desired unstable periodic point. The technique would be useful for experimentalists who want to know the location of the already coexisting unstable point outside chaotic regions. We illustrate the technique using the logistic map. [S1063-651X(96)04410-8]

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I. INTRODUCTION

In the field of nonlinear dynamics, for many years, it had been generally accepted that one could not harness chaotic motions. In 1990, however, Ott, Grebogi, and Yorke suggested a new control method (OGY method) which can convert a chaotic motion to a desired unstable periodic orbit (UPO) within a chaotic attractor by making only small perturbations in an accessible set of system parameters [1,2]. The OGY method has been developed to control various chaotic systems: low-dimensional systems [3], high-dimensional systems [4], time-continuous systems [5,6], and Hamiltonian systems [7]. Several experiments have been performed in various physical systems, including a fluid system [8], a mechanical system [9], electronic systems [10], and laser systems [11]. In addition, other methods based on the following techniques have been proposed: the time delay coordinates [12], neuro-controllers [13,14], the optimal control [15], and the H^∞ control [16]. On the other hand, Pyragas has proposed the *delayed-feedback control* method (DFC method) which does not require *a priori* location of the desired UPO [6]. The DFC method has been developed [17] and applied to physical systems: electronic circuits [18], a laser system [19], and a mechanical system [20].

In a parallel manner, several researchers have proposed the tracking techniques which allow one to follow UPOs over wide ranges of parameters and through bifurcations [21,22]. The tracking techniques are powerful tools for maintaining control of practical chaotic systems influenced by a change in their environment. In addition, for chaotic systems which have a variable system parameter, they are useful for experimentalists who want to know the location or the structure of the desired UPO outside chaotic regions.

For simplicity, let us consider a one-dimensional (1D) nonlinear system which has stable and unstable periodic points. Here we assume that the equation of the nonlinear system is unknown and its parameter p is fixed at a certain value p_0 as shown in Fig. 1. In this situation, an orbit settles on a stable periodic point. We address the following question: Given such a 1D nonlinear system, how can we obtain other periodic motions or know the locations of the already coexisting unstable periodic points by making weak input signals? The methods for controlling chaos and the tracking techniques mentioned above cannot be directly applied to such a nonlinear system. The limiting factor is that the orbit of such a nonlinear system never visits the neighborhood of the unstable periodic point. The purpose of this paper is to offer a technique which overcomes this limitation and solves the above question. The technique uses two delayed-feedback weak signals. The first signal destabilizes the nonlinear system such that the orbit moves from the location of the stable periodic point and then wanders about phase space. The second signal then stabilizes the wandering orbit onto the location of the already coexisting unstable periodic point. Since this technique does not require a large alteration in the system or *a priori* knowledge of the system equation, it would be a powerful tool for experimentalists who want to obtain several periodic motions or know the locations of the already coexisting unstable periodic points in experimental situations.

This paper is organized as follows. Section II describes the 1D nonlinear system to be controlled and explains our

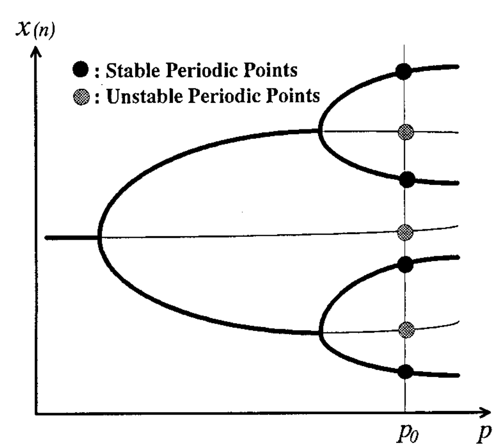

FIG. 1. Schematic illustration of the coexistence stable periodic points and unstable periodic points at the parameter $p=p_0$. Thick lines and thin lines show locations of stable and unstable periodic points at $p\neq p_0$, respectively.

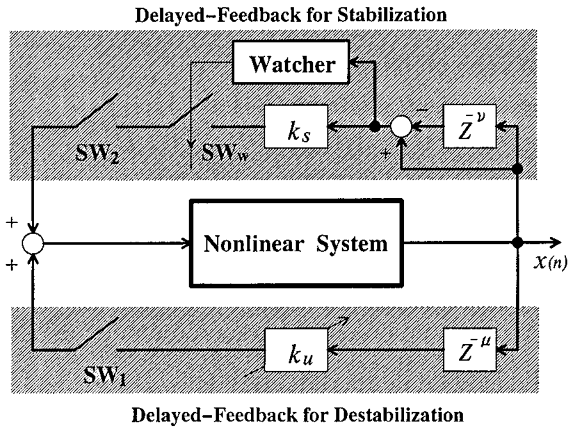

FIG. 2. Control system for realizing the technique.

technique. A brief stability analysis is discussed in Sec. III. In Sec. IV we illustrate the technique using the logistic map. Finally, conclusions of this paper are presented in Sec. V.

II. CONTROL SYSTEM

Let us consider a 1D nonlinear system,

$$x(n+1)=g(x(n),p),\quad x(n)\in\mathbf{R},\quad g:\mathbf{R}\times\mathbf{R}\rightarrow\mathbf{R},\qquad (1)$$

where p is the system parameter. If p is an external variable, we can observe various bifurcation phenomena and chaotic behavior by varying the parameter. This paper, however, assumes that the system parameter p is fixed at a certain value $p=p_0$ as shown in Fig. 1. We describe such nonlinear system as

$$x(n+1)=f(x(n)),\qquad (2)$$

where

$$f(x(n))\triangleq g(x(n),p_0).\qquad (3)$$

This nonlinear system has stable and unstable periodic points. We consider a stable period-q point and an unstable period-ν point,

$$\overline{x_{q,q}}=f(\overline{x_{q,q-1}})=\cdots=f^{q-1}(\overline{x_{q,1}})=f^{q}(\overline{x_{q,q}}),$$

$$\left|\frac{df^{q}(\overline{x_{q,i}})}{dx}\right|<1\quad\forall i(i=1\sim q),$$

$$\hat{x}_{\nu,\nu}=f(\hat{x}_{\nu,\nu-1})=\cdots=f^{\nu-1}(\hat{x}_{\nu,1})=f^{\nu}(\hat{x}_{\nu,\nu}),$$

$$\left|\frac{df^{\nu}(\hat{x}_{\nu,i})}{dx}\right|>1\quad\forall i(i=1\sim\nu),\qquad (4)$$

where $\overline{x_{q,i}}$, $\hat{x}_{\nu,i}$ denote the ith stable period-q point and the ith unstable period-ν point, respectively.

In this nonlinear system an orbit settles on $\overline{x_{q,i}}$. The purpose of the present work is to move the orbit from $\overline{x_{q,i}}$ and then stabilize it onto $\hat{x}_{\nu,i}$. Figure 2 illustrates a control system which realizes the technique. The control system consists of three main parts: the nonlinear system to be controlled, the delayed-feedback signals for destabilizing and stabilizing, and a watcher. The reason we use the delayed-feedback signal to destabilize the nonlinear system is as follows. Since it is well known that the chaotic phenomena can be caused by the delayed-feedback signal in various fields, such as laser systems [23], biology, and physiology [24], we think that, in experimental situations, the use of the delayed-feedback signal is natural and convenient to cause chaotic motions in nonlinear systems without making large alterations. On the other hand, the reason we use the delayed-feedback signal for stabilizing is that we want to use the tracking techniques based on the DFC method which do not require *a priori* location of the desired unstable periodic point [22]. The watcher works not to add the signal to the nonlinear system when the orbit is far from the unstable periodic point $\hat{x}_{\nu,i}$ [14]. The watcher turns on SW_W when the following condition is satisfied:

TABLE I. Operation of our technique.

	Step(0)	Step(1)	Step(2)	Step(3)	Step(4)
SW_1	OFF	ON	ON	ON	OFF
SW_2	OFF	OFF	ON	ON	ON

$$|x(n-\nu)-x(n)|<\epsilon,$$

$$|x(n-j)-x(n)|>\kappa\epsilon,\quad\forall j(j=1\sim\nu-1),\qquad (5)$$

where the small positive ϵ is the watcher threshold and the coefficient κ is set as $\kappa>1$. If we do not use the watcher, a large signal may make the nonlinear system fall into a divergence regime.

The equation of the whole system is given by

$$x(n+1)=f(x(n))+k_u x(n-\mu)+k_s\{x(n-\nu)-x(n)\},\qquad (6)$$

where k_u is the feedback gain for the destabilization and k_s is that for the stabilization. The technique consists of five steps as follows (see Fig. 2 and Table I):

Step 0 The nonlinear system runs freely without any inputs; hence, the orbit settles on the stable period-q point $\overline{x_{q,i}}$;

Step 1 SW_1 is turned on and then the feedback gain k_u varies from 0 to k_{uc} to destabilize the nonlinear system (see Fig. 3);

Step 2 SW_2 is turned on and then the delayed-feedback signal $k_s\{x(n-\nu)-x(n)\}$ stabilizes the orbit onto a location corresponding to the desired unstable period-ν point $\hat{x}_{\nu,i}$ (see Fig. 3);

Step 3 The feedback gain k_u varies from k_{uc} to 0 with a sufficiently slow rate;

Step 4 SW_1 is turned off, but SW_2 keeps turning on.

After Step 4, the orbit is stabilized on the desired unstable period-ν point $\hat{x}_{\nu,i}$.

III. STABILITY ANALYSIS

In this section we shall discuss a stability of the whole system described by Eq. (6). To begin with, we denote a variable state as

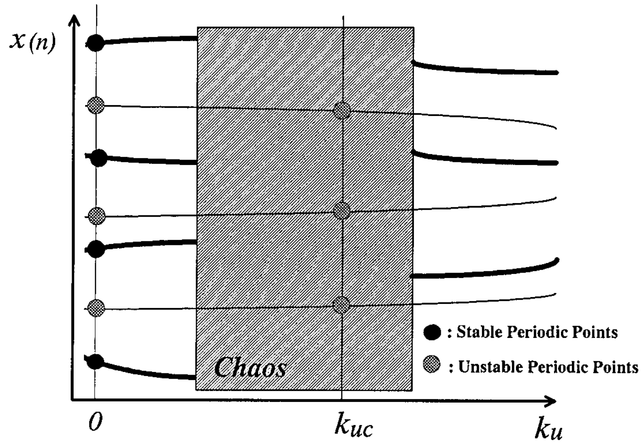

FIG. 3. Schematic illustration of Steps 1∼3. At Step 1, the feedback gain k_u varies slowly from 0 to k_{uc}. At Step 2, it is fixed at $k=k_{uc}$. At Step 3, it varies slowly from k_{uc} to 0.

$$\mathbf{X}_n \triangleq [x(n), x(n-1), \ldots, x(n-\tau)]^T, \tag{7}$$

where

$$\tau = \max(\mu, \nu). \tag{8}$$

From Eq. (7), Eq. (6) can be rewritten as

$$\mathbf{X}_{n+1} = \mathbf{h}(\mathbf{X}_n, k_u) + \mathbf{D}(k_s)\mathbf{X}_n, \tag{9}$$

where

$$\mathbf{h}(\mathbf{X}_n, k_u) = \mathbf{f}(\mathbf{X}_n) + \mathbf{L}(k_u)\mathbf{X}_n,$$

$$\mathbf{f}(\mathbf{X}_n) = [f(x(n)), x(n), x(n-1), \ldots, x(n-\tau+1)]^T,$$

$$\mathbf{L}(k_u) = [l_{ij}] \in \mathbf{R}^{(\tau+1)\times(\tau+1)} \quad l_{ij} = \begin{cases} k_u & i=1, \quad j=\mu+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathbf{D}(k_s) = [d_{ij}] \in \mathbf{R}^{(\tau+1)\times(\tau+1)} \quad d_{ij} = \begin{cases} -k_s & i=1, j=1 \\ +k_s & i=1, j=\nu+1 \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

At Step 1 (i.e., $k_s=0$, $k_u \neq 0$) the stable period-q point in Eq. (4) is rewritten as

$$\overline{\mathbf{X}}_{q,q}(k_u) = \mathbf{h}(\overline{\mathbf{X}}_{q,q-1}(k_u), k_u) = \cdots = \mathbf{h}^{q-1}(\overline{\mathbf{X}}_{q,1}(k_u), k_u) = \mathbf{h}^{q}(\overline{\mathbf{X}}_{q,q}(k_u), k_u). \tag{11}$$

The local linearized matrix at the stable period-q point is

$$\mathbf{H}_{q,i}(k_u) = \left.\frac{\partial \mathbf{h}(\mathbf{X}, k_u)}{\partial \mathbf{X}}\right|_{\mathbf{X}=\overline{\mathbf{X}}_{q,i}(k_u)} \quad (i=1 \sim q). \tag{12}$$

In the neighborhood of $\overline{\mathbf{X}}_{q,i}(k_u)$, the following equation is satisfied:

$$\delta\mathbf{X}_{q+1} = \left[\prod_{j=1}^{q} \mathbf{H}_{q,q+1-j}(k_u)\right] \delta\mathbf{X}_1, \tag{13}$$

where

$$\delta\mathbf{X}_{q+1} = \mathbf{X}_{q+1} - \overline{\mathbf{X}}_{q,1}(k_u), \quad \delta\mathbf{X}_1 = \mathbf{X}_1 - \overline{\mathbf{X}}_{q,1}(k_u). \tag{14}$$

Since the stable period-q point $\overline{\mathbf{X}}_{q,i}(k_u)$ is destabilized at Step 1, the feedback gain k_{uc} and the delay time μ satisfy the following condition:

$$\left|\lambda_i\left[\prod_{j=1}^{q} \mathbf{H}_{q,q+1-j}(k_{uc})\right]\right| > 1, \quad \mathbf{H}_{q,q+1-j} \in \mathbf{R}^{(\tau+1)\times(\tau+1)}, \quad \exists i (i=1 \sim \tau+1), \tag{15}$$

where $\lambda_i[\cdot]$ are the eigenvalues. Note that this condition is necessary to generate the chaotic motion in Eq. (9). However, it is impossible to derive the sufficient condition under which the chaotic motion occurs in Eq. (9); hence, we can not determine the proper μ, k_{uc} in advance. We have to determine the proper μ, k_{uc} by trial-and-error testing.

At Steps 2,3 (i.e., $k_s \neq 0$, $k_u \in [0, k_{uc}]$) we focus on an unstable period-ν point $\mathbf{X}_{\nu,i}(k_u)$,

$$\hat{\mathbf{X}}_{\nu,\nu}(k_u) = \mathbf{h}(\hat{\mathbf{X}}_{\nu,\nu-1}(k_u), k_u) = \cdots = \mathbf{h}^{\nu-1}(\hat{\mathbf{X}}_{\nu,1}(k_u), k_u) = \mathbf{h}^{\nu}(\hat{\mathbf{X}}_{\nu,\nu}(k_u), k_u). \tag{16}$$

The first component of $\hat{\mathbf{X}}_{\nu,i}(0)$ agrees with the desired unstable period-ν point $\hat{x}_{\nu,i}$,

$$\hat{x}^{(1)}_{\nu,i}(0) = \hat{x}_{\nu,i}, \quad \forall i (i=1 \sim \nu), \tag{17}$$

where

$$\hat{\mathbf{X}}_{\nu,i}(0) = [\hat{x}^{(1)}_{\nu,i}(0), \hat{x}^{(2)}_{\nu,i}(0), \ldots, \hat{x}^{(\tau+1)}_{\nu,i}(0)]^T. \tag{18}$$

We can derive the sufficient condition under which the stabilizing and the tracking are achieved. The local linearized matrix at the unstable period-ν point $\hat{\mathbf{X}}_{\nu,i}(k_u)$ is given by

$$\mathbf{H}_{\nu,i}(k_u) = \left.\frac{\partial \mathbf{h}(\mathbf{X}, k_u)}{\partial \mathbf{X}}\right|_{\mathbf{X}=\hat{\mathbf{X}}_{\nu,i}(k_u)} \quad (i=1 \sim \nu). \tag{19}$$

In a neighborhood of $\hat{\mathbf{X}}_{\nu,i}(k_u)$, Eq. (9) is governed by

$$\delta\mathbf{X}_{\nu+1} = \left[\prod_{j=1}^{\nu} [\mathbf{H}_{\nu,\nu+1-j}(k_u) + \mathbf{D}(k_s)]\right] \delta\mathbf{X}_1, \tag{20}$$

where

$$\delta\mathbf{X}_{\nu+1} = \mathbf{X}_{\nu+1} - \hat{\mathbf{X}}_{\nu,1}(k_u), \quad \delta\mathbf{X}_1 = \mathbf{X}_1 - \hat{\mathbf{X}}_{\nu,1}(k_u). \tag{21}$$

From Eq. (20) we obtain the following stability condition:

$$\left|\lambda_i\left[\prod_{j=1}^{\nu} [\mathbf{H}_{\nu,\nu+1-j}(k_u) + \mathbf{D}(k_s)]\right]\right| < 1, \quad \forall k_u \in [0, k_{uc}], \quad \forall i (i=1 \sim \tau+1). \tag{22}$$

If the feedback gain k_s satisfies this condition, then the chaotic orbit can be stabilized and tracked by the delayed-feedback signal $k_s\{x(n-\nu)-x(n)\}$.

IV. CONTROLLING LOGISTIC MAP

In this section we test the technique on numerical experiments. The logistic map is used as the 1D nonlinear system

$$g(x(n),p)=px(n)(1-x(n)), \tag{23}$$

where p is the system parameter. The parameter is fixed at $p=p_0$, where stable and unstable periodic points coexist. The nonlinear system to be controlled is given by

$$f(x(n))=p_0x(n)(1-x(n)). \tag{24}$$

Two examples are given below: one of them is a special case ($\nu=1$, $\mu=1$); another is a general case ($\nu>1$, $\mu>1$).

A. Special case ($\nu=1$, $\mu=1$)

In this case (i.e., $\mu=1$, $\nu=1$), the signal $k_ux(n-1)$ moves the orbit from $\overline{x_{q,i}}$ to $\hat{x}_{1,1}$, and then the signal $k_s\{x(n-1)-x(n)\}$ stabilizes the moving orbit onto $\hat{x}_{1,1}$. It is easy to discuss the stability of the whole system Eq. (6) at Steps 3,4. The functions $\mathbf{h}(\mathbf{X}_n,k_u)$, $\mathbf{D}(k_s)$ in Eq. (10) can be described by

$$\mathbf{h}(\mathbf{X}_n,k_u)=\begin{bmatrix} p_0x(n)(1-x(n))+k_ux(n-1) \\ x(n) \end{bmatrix},$$

$$\mathbf{D}(k_s)=\begin{bmatrix} -k_s & +k_s \\ 0 & 0 \end{bmatrix}. \tag{25}$$

The unstable period-1 point corresponding to the desired $\hat{x}_{1,1}$ is given by $\hat{\mathbf{X}}_{1,1}(k_u)=[(p_0+k_u-1)/p_0,(p_0+k_u-1)/p_0]^T$. The matrix $[\mathbf{H}_{1,1}(k_u)+\mathbf{D}(k_s)]$ is as follows:

$$\mathbf{H}_{1,1}(k_u)+\mathbf{D}(k_s)=\begin{bmatrix} 2-2k_u-p_0-k_s & k_u+k_s \\ 1 & 0 \end{bmatrix}. \tag{26}$$

To verify the relation between k_u and k_s, we change this matrix to the following polynomial:

$$z^2+(2k_u+p_0+k_s-2)z-(k_u+k_s)=0. \tag{27}$$

The stability condition of this polynomial is

$$|k_u+k_s|<1,\ \ k_u-1+p_0>0, \quad -2k_s-3k_u-p_0+3>0. \tag{28}$$

Figure 4 shows the stable region on the k_u-k_s plane. The arrows in Fig. 4 correspond to Steps 1~3. The gain k_{uc} is allowed to be in a range $((3-p_0)/3,(5-p_0)/3)$. The gain k_s is chosen in a range $(-1,(3-p_0)/2)$ for $k_{uc}>0$ or $(-k_{uc}-1,(3-p_0)/2)$ for $k_{uc}<0$.

Figure 5 shows the numerical result. The parameter p is fixed at $p_0=3.50$, where the stable period-4 point ($\overline{x_{4,i}}$), the unstable period-1 point ($\hat{x}_{1,1}$), and the unstable period-2 point ($\hat{x}_{2,i}$) coexist. At Step 0, the orbit settles on $\overline{x_{4,i}}$. At Step 1, SW_1 is turned on and then the feedback gain k_u varies slowly from 0 to $k_{uc}=0.08$ with $k_u=k_{uc}(n-500)/500$ for $n\in[500,1000]$. At Step 2, SW_2 is turned on and then the stabilization of the unstable period-1 point $\hat{\mathbf{X}}_{1,1}(k_{uc})$ is achieved with $k_s=-0.75$. At Step 3, the feedback gain k_u varies slowly from $k_{uc}=0.08$ to 0 with $k_u=k_{uc}\{1-(n-1500)/500\}$ for $n\in[1500,2000]$. At Step 4, SW_1 is turned off. After all steps have been done, the stabilization of the desired unstable period-1 point $\hat{x}_{1,1}$ is achieved successfully.

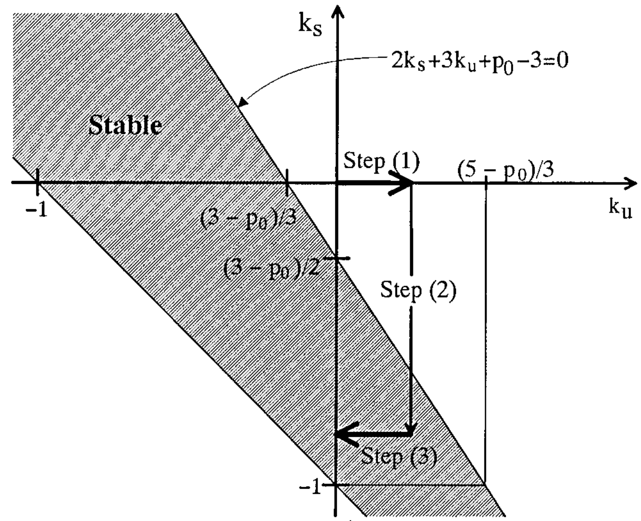

FIG. 4. Stable region on k_u-k_s plane for the special case ($\nu=1$, $\mu=1$): The logistic map. Three arrows correspond to Steps 1~3. Thick arrows represent that the gain k_u varies slowly at Steps 1,3, and a thin arrow shows that the gain k_s switches from 0 to k_s at Step 2.

B. General case ($\nu>1$, $\mu>1$)

In the general case ($\nu>1$, $\mu>1$), it is not easy to discuss the stability of the whole system Eq. (6) because the relation between k_u and k_s is not simple. However, adjusting the gain

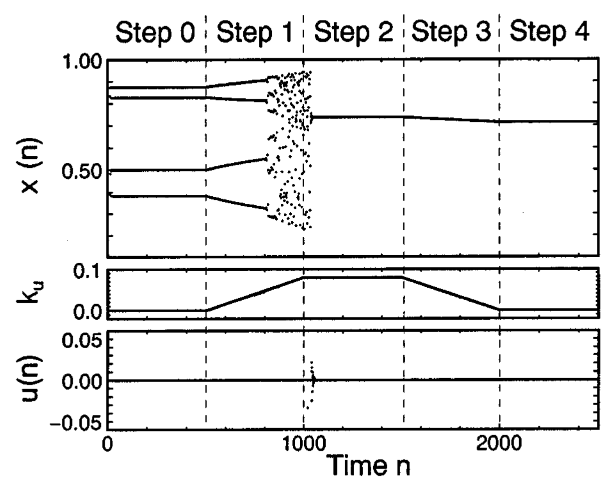

FIG. 5. Stabilizing the unstable period-1 point of the logistic map in period-4 stable state. In this experiment the following parameters are used: $p_0=0.35$, $\nu=1$, $\mu=1$, $k_{uc}=0.08$, $k_s=-0.75$, $\epsilon=0.05$, $\kappa=5$.

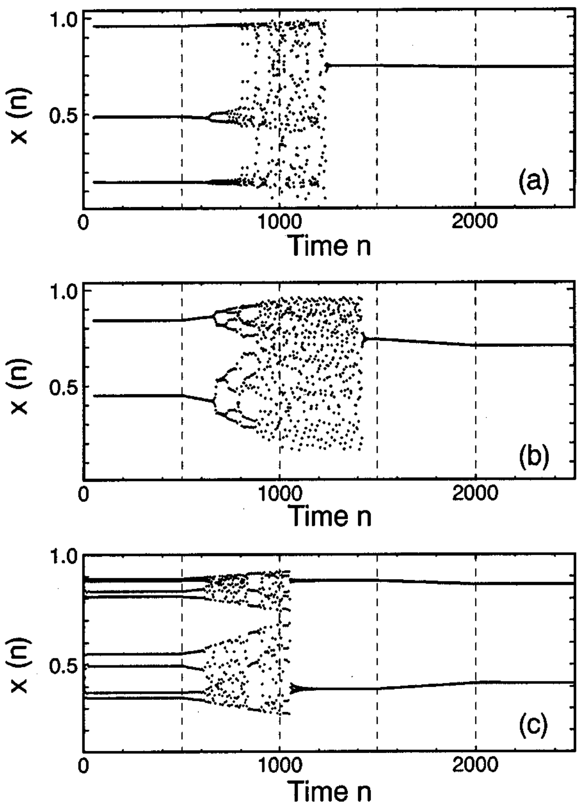

FIG. 6. Stabilizing unstable periodic points of the logistic map. (a) From the stable period-3 point $\overline{x_{3,i}}$ to the unstable period-1 point $\hat{x}_{1,1}$: $p_0=3.84$, $k_s=-0.75$, $k_{uc}=0.03$, $\mu=3$, $\nu=1$, $\epsilon=0.05$, $\kappa=5$. (b) From the stable period-2 point $\overline{x_{2,i}}$ to the unstable period-1 point $\hat{x}_{1,1}$: $p_0=3.40$, $k_s=-0.75$, $k_{uc}=0.12$, $\mu=3$, $\nu=1$, $\epsilon=0.05$, $\kappa=5$. (c) From the stable period-8 point $\overline{x_{8,i}}$ to the unstable period-2 point $\hat{x}_{2,i}$: $p_0=3.56$, $k_s=0.3$, $k_{uc}=0.04$, $\mu=1$, $\nu=2$, $\epsilon=0.05$, $\kappa=5$.

k_s, k_{uc} to proper values by trial-and-error testing, we can achieve the stabilization of the desired unstable periodic point. Figure 6 shows numerical experiments for $\overline{x_{3,i}}\rightarrow\hat{x}_{1,1}$ at $p_0=3.84$, $\overline{x_{2,i}}\rightarrow\hat{x}_{1,1}$ at $p_0=3.40$, and $\overline{x_{8,i}}\rightarrow\hat{x}_{2,i}$ at $p_0=3.56$. The feedback gain k_u varies as $k_u=k_{uc}(n-500)/500$ for $n\in[500,1000]$, $k_u=k_{uc}\{1-(n-1500)/500\}$ for $n\in[1500,2000]$.

In addition, once we find the proper gain k_{uc} and delay time μ which cause the chaotic motion, the switching operation among several unstable periodic points within the chaotic attractor can be achieved easily by only changing the period ν of the watcher and tuning the feedback gain k_s. Figure 7 shows the result of our numerical switching operation from the stable period-4 point to the unstable period-1, -2 points ($\overline{x_{4,i}}\rightarrow\hat{x}_{1,1}\rightarrow\hat{x}_{2,i}\rightarrow\overline{x_{4,i}}$).

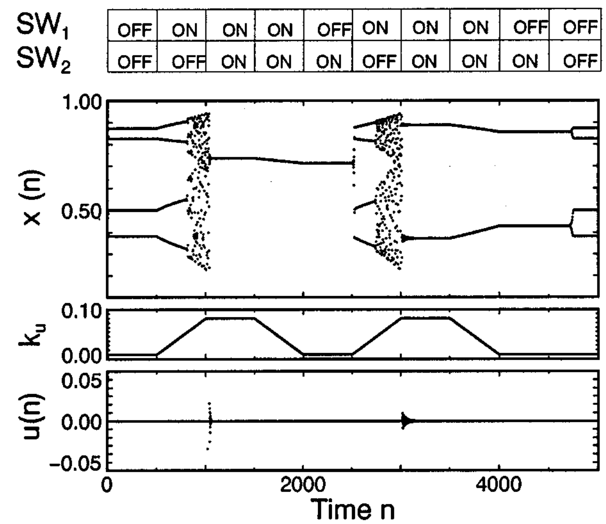

FIG. 7. Switching unstable period-1, -2 points of the logistic map in period-4 stable state. In this experiment the following parameters are used: $p_0=0.35$, $\nu=1,2$, $\mu=1$, $k_{uc}=0.08$, $\epsilon=0.05$, $\kappa=5$, $k_s=-0.75$ for period-1, $k_s=0.30$ for period-2.

V. CONCLUSIONS

We have proposed a technique to realize the following operation: an orbit in a stable 1D nonlinear system stabilizes on an already coexisting unstable periodic point. The advantages of the technique may be summarized as follows. (i) Once the proper gain k_{uc} and delay time μ are found by a trial-and-error testing, one can obtain several periodic motions by changing the period ν and tuning the gain k_s. (ii) The technique allows experimentalists to know the locations of the already coexisting unstable periodic points outside chaotic regions. (iii) It does not require us to make a large alteration in the nonlinear system. (iv) It does not require the equation of the nonlinear system. On the other hand, the main disadvantage of the technique is that it can not be applied to all nonlinear systems. The reason is that there exist nonlinear systems which never generate chaotic motions by using any delayed-feedback signals. We can not discriminate the available nonlinear systems, since it is impossible to derive the condition under which the delayed-feedback signal generates chaotic motions.

Although this paper has dealt with time-discrete one-dimensional nonlinear systems, the technique may have the ability to control high-dimensional or time-continuous nonlinear systems. Therefore, further work is to develop the technique for control of high-dimensional nonlinear systems. In addition, we think that the technique should be implemented to test its performance on experimental situations by using simple electronic, laser, or mechanical nonlinear systems.

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